HYDRODYNAMIC FLOW STABILITY OF A FLUID WITH A POWER-LAW RHEOLOGICAL BEHAVIOR IN A CHANNEL WITH ELASTIC WALLS

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The stability of steady gradiental flow of a non-Newtonian fluid with a power-law rheological behavior in a channel with elastic walls is analyzed.

With regard to certain problems in biomechanics and polymer physics, the authors of [1, 2] have studied the hydrodynamic stability of steady flow modes of non-Newtonian fluids. They considered the stability of flow in channels with undeformable walls. It would be of interest to consider the stability of flow of such fluids in channels with elastic walls and to analyze how the critical Reynolds number is affected by the elastic properties of such walls.

We consider the gradiental flow of a fluid with a power-law rheological behavior in a flat channel. The distribution of the dimensionless velocity is in this case [3]

$$U(y) = 1 - |y|^{\frac{n+1}{n}} \quad (-1 \leqslant y \leqslant 1), \tag{1}$$

with n denoting the rheological parameter of the system. The problem of stability here in response to infinitesimally small two-dimensional perturbations reduces to that of universal Orr–Sommerfeld equation [1].

The first pair of linearly independent solutions $\psi_{1,2}$ is sought in the form of series, following the general procedure [1, 4], with the aid of the recurrence relations for the coefficients [5]. The second pair of linearly independent terminating solutions $\psi_{3,4}$ is

$$\psi_{3,4} = (U-c)(DU)^{\frac{n-1}{4}} \exp\left[\pm \int\limits_{y_c}^{y} \sqrt{\frac{i\alpha \operatorname{Re}(U-c)}{n(DU)^{n-1}}}\, dy\right]. \tag{2}$$

Here y_c denotes the critical point where the flow velocity is equal to the perturbation velocity and $\mathrm{Re} = \rho U_0^{2-n} L^n / k_n$ is the universal Reynolds number.

The general solution to the universal Orr–Sommerfeld equation will be sought as a superposition of the linearly independent solutions:

$$\psi(y) = \sum_{i=1}^{4} C_i \psi_i, \tag{3}$$

with the arbitrary constants C_i.

In order to find the eigenvalues of the Orr–Sommerfeld equation, it is necessary to establish the boundary conditions of the problem. We will consider only normal strains of the channel walls, since calculations have shown that tangential strains have a negligible effect on the flow stability – just as in the case of a Newtonian fluid [6].

In the case of small normal displacements, the strain–stress relation for an elastic wall can be written as follows:

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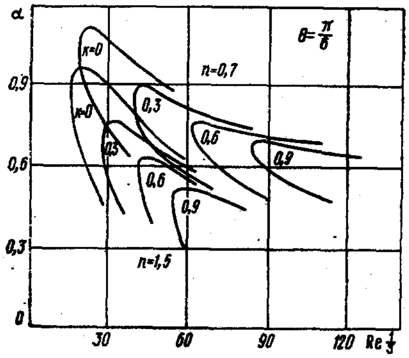

Fig. 1

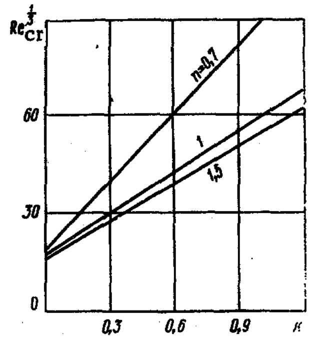

Fig. 2

Fig. 1. Curves of neutral stability for n = 0.7 and 1.5 at $\theta = \pi/6$ and with k = 0, 0.3, 0.6, 0.9.

Fig. 2. Relation between $\mathrm{Re}_{cr}^{1/3}$ and parameter k for n = 0.7, 1.0, 1.5 at $\theta = \pi/6$.

$$\zeta = kpe^{i\theta}. \tag{4}$$

Here ζ denotes the normal displacement of a wall surface.

Differentiating (4) with respect to time and expressing all quantities in terms of perturbations, with the equation of flow properly taken into account, we obtain the following boundary conditions at a wall:

$$\begin{gathered} -\frac{kcn\,(DU)^{n-1}}{i\alpha\,\mathrm{Re}}\psi'''(-1) + [e^{-i\theta} - kcDU(-1)]\,\psi(-1) \cdot \\ - D^2U\,(DU)^{n-2}\,\frac{kcn\,(n-1)}{i\alpha\,\mathrm{Re}}\,[\psi''(-1) + \alpha^2\psi(-1)] = 0, \\ \psi'(-1) = 0. \end{gathered} \tag{5}$$

It is to be noted that, when n = 1, expressions (5) become the corresponding boundary conditions for a Newtonian fluid [6].

The second pair of boundary conditions, at the center of the channel y = 0, needed for solving the universal Orr–Sommerfeld equation are found as follows [5].

For dilatant fluid (n > 1) DU(0) = 0. Therefore, the terminating solutions $\psi_{3,4}(0) = 0$ and their derivatives $D\psi_{3,4}(0)$ are singular. The singularity in the derivative of the general solution (3) at point y = 0 is removable, if the condition

$$C_3D\psi_3(0) + C_4D\psi_4(0) = 0. \tag{6}$$

is satisfied.

The condition of even-numbered perturbations at point y = 0

$$\sum_{i=1}^{4} C_iD\psi_i = 0 \tag{7}$$

together with conditions (5) and (6), yield a secular equation which, with all terms ranked according to their orders of magnitude, can be written as

$$\frac{\psi_3(-1)}{D\psi_3(-1)} = \frac{\begin{vmatrix}\psi_1(-1) & \psi_2(-1)\\ D\psi_1(0) & D\psi_2(0)\end{vmatrix}}{\begin{vmatrix}D\psi_1(-1) & D\psi_2(-1)\\ D\psi_1(0) & D\psi_2(0)\end{vmatrix}} - \frac{kc^2}{[e^{-i\theta} - kcDU(-1)]}. \tag{8}$$

The left-hand side of Eq. (8) can be expressed in terms of the tabulated Titjens function [4]; the right-hand side of Eq. (8) can be evaluated on the basis of the solutions given here earlier.

A stability analysis of pseudoplastic fluids can, in the final count, also be reduced to finding the eigenvalues of the secular equation (8).

Some results of calculations made for walls with compliance in the normal direction are shown in Figs. 1 and 2.

The curves in Fig. 1 depict the neutral stability for n = 0.7 and 1.5, at various values of the parameter k with $\theta = \pi/6$.

The curves in Fig. 2 depict the universal critical Reynolds number as a function of k, for n = 0.7, 1.0, 1.5 and $\theta = \pi/6$. At a given phase shift, according to this graph, the universal critical Reynolds number is a monotonic function of k. Such a trend is explainable in terms of the energy flux transmitted from the mainstream to an elastic wall. This flux is $W = -\overline{p\tilde{u}} \sim kp^2 \sin\theta$, i.e., proportional to k and, therefore, there must be some monotonic relation between the flow stability and the parameter k. For a given value of k, moreover, the critical Reynolds number decreases with higher values of n. This can be explained as follows. As is well known, the flow stability depends on the shape of the velocity profile and thus, in the final analysis, the Reynolds stresses influence the mechanism of energy transfer from the mainstream to the perturbations. A flatter velocity profile will be more stable [7, 8]. Since the velocity profile (1) becomes flatter for lower values of n, hence one should obviously expect the critical Reynolds number to become higher as the value of n decreases (with all other conditions unchanged).

NOTATION

n and k_n	are the rheological parameters of a fluid;
x and y	are Cartesian coordinates;
U(y)	is the velocity profile;
$D \equiv d/dy$	is the differential operator;
α	is the wave number;
c	is the velocity of perturbation propagation;
L	is the channel half-width;
U_0	is the characteristic velocity;
$\psi(y)$	is the amplitude of the flow-perturbation function;
p	is the variable component of pressure on a wall;
k	is a parameter which characterizes the elastic properties of a wall;
θ	is the phase shift between stresses and strains in a wall.

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